

# CIS 300: Foundations of Computer Science

## Looking at Pascal's Triangle

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# Drawing Pascal's Triangle

# Filling Pascal's Triangle

# Navigating Pascal's Triangle

## Parallel Row

Z

	0	1	2	3	4	5	6	7	8	9
0	$\Gamma$ 1	$\sigma$ 1	$\pi$ 1	$\lambda$ 1	$\mu$ 1	$\delta$ 1	$\zeta$ 1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	$S$ 5	$N$ 6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	$\omega$ 10	$\xi$ 15	21	28	36		
3	$D$ 1	$E$ 4	$F$ 10	$P$ 20	$Y$ 35	56	84			
4	$H$ 1	$M$ 5	$K$ 15	35	70	126				
5	$P$ 1	$Q$ 6	21	56	126					
6	$V$ 1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

# Navigating Pascal's Triangle

## Perpendicular Row

Z

	0	1	2	3	4	5	6	7	8	9
0	$\Gamma$ 1	$\sigma$ 1	$\pi$ 1	$\lambda$ 1	$\mu$ 1	$\delta$ 1	$\zeta$ 1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	$S$ 5	$N$ 6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	$\omega$ 10	$\xi$ 15	21	28	36		
3	$D$ 1	$E$ 4	$F$ 10	$P$ 20	$Y$ 35	56	84			
4	$H$ 1	$M$ 5	$K$ 15	35	70	126				
5	$P$ 1	$Q$ 6	21	56	126					
6	$V$ 1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

# Navigating Pascal's Triangle

Base

Z

	0	1	2	3	4	5	6	7	8	9
0	$\Gamma$ 1	$\sigma$ 1	$\pi$ 1	$\lambda$ 1	$\mu$ 1	$\delta$ 1	$\zeta$ 1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	$S$ 5	$N$ 6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	$\omega$ 10	$\xi$ 15	21	28	36		
3	$D$ 1	$E$ 4	$F$ 10	$P$ 20	$Y$ 35	56	84			
4	$H$ 1	$M$ 5	$K$ 15	35	70	126				
5	$P$ 1	$Q$ 6	21	56	126					
6	$V$ 1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

# Navigating Pascal's Triangle

## Reciprocal

Z

	0	1	2	3	4	5	6	7	8	9
0	1 <sup>σ</sup>	1 <sup>π</sup>	1 <sup>λ</sup>	1 <sup>μ</sup>	1 <sup>δ</sup>	1 <sup>ζ</sup>	1	1	1	
1	1 <sup>φ</sup>	2 <sup>θ</sup>	3 <sup>ρ</sup>	4 <sup>ς</sup>	5 <sup>ν</sup>	6	7	8	9	
2	1 <sup>α</sup>	3 <sup>β</sup>	6 <sup>ε</sup>	10 <sup>ω</sup>	15 <sup>ξ</sup>	21	28	36		
3	1 <sup>δ</sup>	4 <sup>ε</sup>	10 <sup>φ</sup>	20 <sup>χ</sup>	35 <sup>ψ</sup>	56	84			
4	1 <sup>η</sup>	5 <sup>ι</sup>	15 <sup>κ</sup>	35 <sup>λ</sup>	70	126				
5	1 <sup>π</sup>	6 <sup>ρ</sup>	21 <sup>σ</sup>	56 <sup>τ</sup>	126					
6	1 <sup>υ</sup>	7	28	84						
7	1	8	36							
8	1	9								
9	1									

# Pascal's Triangle

## Modern Notation

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$



# Pascal's Triangle

## Modern Notation - Row

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$   
Row:

# Pascal's Triangle

## Modern Notation - Row

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$   
Row: fixed row index.

# Pascal's Triangle

## Modern Notation - Column

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$

Row: fixed row index.

Col:

# Pascal's Triangle

## Modern Notation - Column

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$

Row: fixed *row* index.

Col: fixed *column* index.

# Pascal's Triangle

## Modern Notation - Base

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$

Row: fixed *row* index.

Col: fixed *column* index.

Base:

# Pascal's Triangle

## Modern Notation - Base

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$

Row: fixed *row* index.

Col: fixed *column* index.

Base: fixed *sum* of row and column indices.

# Pascal's Triangle

## Modern Notation - Reciprocal

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$

Row: fixed *row* index.

Col: fixed *column* index.

Base: fixed *sum* of row and column indices.

Reciprocals:

# Pascal's Triangle

## Modern Notation - Reciprocal

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

$T_{\text{row},\text{column}}$

Row: fixed *row* index.

Col: fixed *column* index.

Base: fixed *sum* of row and column indices.

Reciprocals:  $T_{r,c}$  and  $T_{c,r}$ .



# Pascal's Triangle

## Modern Notation - Filling the Cells

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

Recurrence:

$$T_{r,c} = T_{(r-1),c} + T_{r,(c-1)}$$

# Pascal's Triangle

## Modern Notation - Filling the Cells

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

Recurrence:

$$T_{r,0} = T_{(r-1),0} + 0$$

$$T_{r,c} = T_{(r-1),c} + T_{r,(c-1)}$$

# Pascal's Triangle

## Modern Notation - Filling the Cells

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

Recurrence:

$$T_{0,c} = 0 + T_{0,(c-1)}$$

$$T_{r,0} = T_{(r-1),0} + 0$$

$$T_{r,c} = T_{(r-1),c} + T_{r,(c-1)}$$

# Pascal's Triangle

## Modern Notation - Filling the Cells

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

Recurrence:

$$T_{0,0} = G \text{ (generator value)}$$

$$T_{0,c} = 0 + T_{0,(c-1)}$$

$$T_{r,0} = T_{(r-1),0} + 0$$

$$T_{r,c} = T_{(r-1),c} + T_{r,(c-1)}$$

# Pascal's Triangle

## The First Consequence

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$ 1	$T_{0,1}$ 1	$T_{0,2}$ 1	$T_{0,3}$ 1	$T_{0,4}$ 1	$T_{0,5}$ 1	$T_{0,6}$ 1	$T_{0,7}$ 1	$T_{0,8}$ 1	$T_{0,9}$ 1
1	$T_{1,0}$ 1	$T_{1,1}$ 2	$T_{1,2}$ 3	$T_{1,3}$ 4	$T_{1,4}$ 5	$T_{1,5}$ 6	$T_{1,6}$ 7	$T_{1,7}$ 8	$T_{1,8}$ 9	
2	$T_{2,0}$ 1	$T_{2,1}$ 3	$T_{2,2}$ 6	$T_{2,3}$ 10	$T_{2,4}$ 15	$T_{2,5}$ 21	$T_{2,6}$ 28	$T_{2,7}$ 36		
3	$T_{3,0}$ 1	$T_{3,1}$ 4	$T_{3,2}$ 10	$T_{3,3}$ 20	$T_{3,4}$ 35	$T_{3,5}$ 56	$T_{3,6}$ 84			
4	$T_{4,0}$ 1	$T_{4,1}$ 5	$T_{4,2}$ 15	$T_{4,3}$ 35	$T_{4,4}$ 70	$T_{4,5}$ 126				
5	$T_{5,0}$ 1	$T_{5,1}$ 6	$T_{5,2}$ 21	$T_{5,3}$ 56	$T_{5,4}$ 126					
6	$T_{6,0}$ 1	$T_{6,1}$ 7	$T_{6,2}$ 28	$T_{6,3}$ 84						
7	$T_{7,0}$ 1	$T_{7,1}$ 8	$T_{7,2}$ 36							
8	$T_{8,0}$ 1	$T_{8,1}$ 9								
9	$T_{9,0}$ 1									

*In every arithmetical triangle all the cells of the first parallel row and of the first perpendicular row are the same as the generating cell.*

# Pascal's Triangle

## The First Consequence

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

*In every arithmetical triangle all the cells of the first parallel row and of the first perpendicular row are the same as the generating cell.*

$$\forall r, c \in \mathbb{Z}^{\geq 0}, T_{0,c} = T_{r,0} = G.$$

# Pascal's Triangle

## The First Consequence

	0	1	2	3	4	5	6	7	8	9
0	$\overset{G}{T_{0,0}}$ 1	$\overset{\sigma}{T_{0,1}}$ 1	$\overset{\pi}{T_{0,2}}$ 1	$T_{0,3}$ 1	$T_{0,4}$ 1	$T_{0,5}$ 1	$T_{0,6}$ 1	$T_{0,7}$ 1	$T_{0,8}$ 1	$T_{0,9}$ 1
1	$\overset{\phi}{T_{1,0}}$ 1	$T_{1,1}$ 2	$T_{1,2}$ 3	$T_{1,3}$ 4	$T_{1,4}$ 5	$T_{1,5}$ 6	$T_{1,6}$ 7	$T_{1,7}$ 8	$T_{1,8}$ 9	
2	$\overset{A}{T_{2,0}}$ 1	$T_{2,1}$ 3	$T_{2,2}$ 6	$T_{2,3}$ 10	$T_{2,4}$ 15	$T_{2,5}$ 21	$T_{2,6}$ 28	$T_{2,7}$ 36		
3	$T_{3,0}$ 1	$T_{3,1}$ 4	$T_{3,2}$ 10	$T_{3,3}$ 20	$T_{3,4}$ 35	$T_{3,5}$ 56	$T_{3,6}$ 84			
4	$T_{4,0}$ 1	$T_{4,1}$ 5	$T_{4,2}$ 15	$T_{4,3}$ 35	$T_{4,4}$ 70	$T_{4,5}$ 126				
5	$T_{5,0}$ 1	$T_{5,1}$ 6	$T_{5,2}$ 21	$T_{5,3}$ 56	$T_{5,4}$ 126					
6	$T_{6,0}$ 1	$T_{6,1}$ 7	$T_{6,2}$ 28	$T_{6,3}$ 84						
7	$T_{7,0}$ 1	$T_{7,1}$ 8	$T_{7,2}$ 36							
8	$T_{8,0}$ 1	$T_{8,1}$ 9								
9	$T_{9,0}$ 1									

$$\forall r, c \in \mathbb{Z}^{\geq 0}, T_{0,c} = T_{r,0} = G.$$

For by definition each cell of the triangle is equal to the sum of the immediately preceding perpendicular and parallel cells. But the cells of the first parallel row have no preceding perpendicular cells, and those of the first perpendicular row have no preceding parallel cells; therefore they are all equal to each other and consequently to the generating number.

Thus  $\varphi = G + 0$ , that is,  $\varphi = G$ ,

$A = \varphi + 0$ , that is,  $\varphi$ ,

$\sigma = G + 0$ ,  $\pi = \sigma + 0$ ,

And similarly of the rest.

# Pascal's Triangle

## The First Consequence

	0	1	2	3	4	5	6	7	8	9
0	$T_{0,0}$	$T_{0,1}$	$T_{0,2}$	$T_{0,3}$	$T_{0,4}$	$T_{0,5}$	$T_{0,6}$	$T_{0,7}$	$T_{0,8}$	$T_{0,9}$
1	$T_{1,0}$	$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{1,4}$	$T_{1,5}$	$T_{1,6}$	$T_{1,7}$	$T_{1,8}$	
2	$T_{2,0}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{2,4}$	$T_{2,5}$	$T_{2,6}$	$T_{2,7}$		
3	$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{3,4}$	$T_{3,5}$	$T_{3,6}$			
4	$T_{4,0}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$	$T_{4,5}$				
5	$T_{5,0}$	$T_{5,1}$	$T_{5,2}$	$T_{5,3}$	$T_{5,4}$					
6	$T_{6,0}$	$T_{6,1}$	$T_{6,2}$	$T_{6,3}$						
7	$T_{7,0}$	$T_{7,1}$	$T_{7,2}$							
8	$T_{8,0}$	$T_{8,1}$								
9	$T_{9,0}$									

TBP:  $\forall r, c \in \mathbb{Z}^{\geq 0}, T_{0,c} = T_{r,0} = G$ .

By definition:

$$\forall c > 0 T_{0,c} = T_{0,(c-1)};$$

$$\forall r > 0 T_{r,0} = T_{(r-1),0}$$

$$T_{0,0} = G$$

Thus

$$T_{1,0} = T_{0,0} + 0 = G \quad T_{0,1} = T_{0,0} + 0 = G$$

$$T_{2,0} = T_{1,0} + 0 = G \quad T_{0,2} = T_{0,1} + 0 = G$$

$$T_{3,0} = T_{2,0} + 0 = G \quad T_{0,3} = T_{0,2} + 0 = G$$

$$\therefore \forall r, c \in \mathbb{Z}^{\geq 0}, T_{0,c} = T_{r,0} = G.$$



# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	
2	1	3	6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

*In every arithmetical triangle each cell is equal to the sum of all the cells of the preceding parallel row from its own perpendicular row to the first, inclusive.*

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	
2	1	3	6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$

*In every arithmetical triangle each cell is equal to the sum of all the cells of the preceding parallel row from its own perpendicular row to the first, inclusive.*

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	5	6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$

Let any cell,  $\omega$ , be taken. I say that it is equal to  $R + \theta + \psi + \phi$ , which are the cells of the next higher parallel row from the perpendicular row of  $\omega$  to the first perpendicular row.

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	5	6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$

$$\omega = R + C$$

$\underbrace{\hspace{1.5cm}}_{\theta + B}$

$\underbrace{\hspace{1.5cm}}_{\psi + A}$

$\underbrace{\hspace{1.5cm}}_{\phi}$

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	5	6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$

$$\omega = R + C$$

$\underbrace{\hspace{2em}}_{\theta + B}$

$\underbrace{\hspace{2em}}_{\psi + \phi}$

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	5	6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$
$$\omega = R + C$$

$\underbrace{\hspace{10em}}_{\theta + \psi + \phi}$

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$R$ 4	5	6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$
$$\omega = R + \theta + \psi + \phi$$

# Pascal's Triangle

## The Second Consequence

	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	$\phi$ 1	$\psi$ 2	$\theta$ 3	$\rho$ 4	5	6	7	8	9	
2	$A$ 1	$B$ 3	$C$ 6	10	15	21	28	36		
3	1	4	10	20	35	56	84			
4	1	5	15	35	70	126				
5	1	6	21	56	126					
6	1	7	28	84						
7	1	8	36							
8	1	9								
9	1									

$$\text{TBP: } \forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$

$$T_{2,3} = T_{1,3} + T_{2,2}$$

Def.

$$T_{2,3} = T_{1,3} + T_{1,2} + T_{2,1}$$

Def.

$$T_{2,3} = T_{1,3} + T_{1,2} + T_{1,1} + T_{2,0}$$

Def.

$$T_{2,3} = T_{1,3} + T_{1,2} + T_{1,1} + T_{1,0}$$

Csq 1.

$$T_{2,3} = T_{1,0} + T_{1,1} + T_{1,2} + T_{1,3}$$

Assoc.

$$\therefore T_{2,3} = \sum_{i=0}^3 T_{1,i}$$

$$\therefore \forall r \in \mathbb{Z}^+; T_{r,c} = \sum_{i=0}^c T_{(r-1),i}$$