Basic Structures: Sets, Functions, Sequences, and Series

Set Operations

February 22, 2016
Outline

1. Set Operations
   - Combining Operations
Set Operations

Union

Definition

Let $A$ and $B$ be sets. The union of sets $A$ and $B$, denoted $A \cup B$ ($\cup$), contains all elements that are in $A$ or $B$ (inclusive or).

Set Builder Notation:

$$A \cup B = \{x | x \in A \lor x \in B\}$$
Set Operations

Union

**Definition**

Let $A$ and $B$ be sets. The **union** of sets $A$ and $B$, denoted $A \cup B$ ($\cup$), contains all elements that are in $A$ or $B$ (inclusive or).

**Set Builder Notation:**

$A \cup B = \{ x \mid x \in A \vee x \in B \}$

**Example**

$\{2, 3, 8\} \cup \{3, 5, 8, 9\} = \{2, 3, 5, 8, 9\}$. 
Set Operations

Intersection

**Definition**

Let $A$ and $B$ be sets. The intersection of sets $A$ and $B$, denoted $A \cap B$ ($\cap$), contains all elements that are in both $A$ and $B$.

**Set Builder Notation:**

$A \cap B = \{ x | x \in A \land x \in B \}$
Set Operations

Intersection

Definition
Let $A$ and $B$ be sets. The **intersection** of sets $A$ and $B$, denoted $A \cap B$ (\(\cap\)), contains all elements that are in both $A$ and $B$.

Set Builder Notation:
\[
A \cap B = \{x | x \in A \land x \in B\}
\]

Example
\[
\{2, 3, 8\} \cap \{3, 5, 8, 9\} = \{3, 8\}.
\]
Set Operations

Intersection

**Definition**

*A and B are disjoint* if their intersection is empty.

Note:

\[ |A \cup B| = |A| + |B| - |A \setminus B| \]

Example:

\[ A = \{1, 3, 5\} \text{ and } B = \{2, 4, 6\} \text{ are disjoint because } A \setminus B = \emptyset. \]
Set Operations

Intersection

Definition

A and B are disjoint if their intersection is empty.

Example

\( A = \{1, 3, 5\} \) and \( B = \{2, 4, 6\} \) are disjoint because \( A \cap B = \emptyset \).
Definition

A and B are disjoint if their intersection is empty.

**Note:** \(|A \cup B| =

Example

A = \{1, 3, 5\} and B = \{2, 4, 6\} are disjoint because \(A \cap B = \emptyset\).
Set Operations

Intersection

Definition

A and B are **disjoint** if their intersection is empty.

**Note:** \(|A \cup B| = |A| + |B| - |A \cap B|\)

Example

A = \{1, 3, 5\} and B = \{2, 4, 6\} are disjoint because \(A \cap B = \emptyset\).
Set Operations

Difference

Definition
Let $A$ and $B$ be sets. The *difference* of sets $A$ and $B$, denoted $A - B$, contains all elements that are in $A$, but not $B$.

Set Builder Notation:

$$A - B = \{ x \mid x \in A \land x \notin B \}$$
Set Operations

Difference

Definition
Let $A$ and $B$ be sets. The \textit{difference} of sets $A$ and $B$, denoted $A - B$, contains all elements that are in $A$, but not $B$.

Set Builder Notation:

$$A - B = \{x | x \in A \land x \notin B\}$$

Example

$$\{2, 3, 8\} - \{3, 5, 8, 9\} = \{2\}.$$
Set Operations

Compliment

**Definition**

The universal set, $U$, contains all the elements in the domain. Let $A$ be a set drawn from universe $U$. The *compliment* of set $A$, denoted $\overline{A}$, is $U - A$.

**Set Builder Notation:**

$$\overline{A} = \{x \mid x \notin A\}$$
Set Operations

Compliment

Definition

The universal set, $U$, contains all the elements in the domain. Let $A$ be a set drawn from universe $U$. The compliment of set $A$, denoted $\overline{A}$, ($\overline{\{A\}}$) is $U - A$.

Set Builder Notation:

$\overline{A} = \{x | x \not\in A\}$

Example

Let $U = \mathbb{R}$ and $A = \mathbb{Q}$. Then $\overline{A}$ is all irrational numbers.
Set Operations

Compliment

Definition

The universal set, $U$, contains all the elements in the domain. Let $A$ be a set drawn from universe $U$. The compliment of set $A$, denoted $\overline{A}$, is $U - A$. Set Builder Notation:

$$\overline{A} = \{x|x \not\in A\}$$

Example

Let $U = \mathbb{R}$ and $A = \mathbb{Q}$. Then $\overline{A}$ is all irrational numbers.

Let $U = \mathbb{Z}^+$ and let $A = \{x|x > 10\}$. Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Set Identities

Identity Laws

The Identity Laws
\[ A \cup \emptyset = A \]
\[ A \cap U = A \]

The Domination Laws
\[ A \cup U = U \]
\[ A \cap \emptyset = \emptyset \]

The Idempotent Laws
\[ A \cup A = A \]
\[ A \cap A = A \]

The Double Compliment Law
\[ \overline{(\overline{A})} = A \]
The Commutative Laws
\[ A \cup B = B \cup A \]
\[ A \cap B = B \cap A \]

The Associative Laws
\[ A \cup (B \cup C) = (A \cup B) \cup C \]
\[ A \cap (B \cap C) = (A \cap B) \cap C \]

The Distributive Laws
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
Set Identities

DeMorgan’s and More

De Morgan’s Laws
\[ A \cup B = \overline{A} \cap \overline{B} \]
\[ A \cap B = \overline{A} \cup \overline{B} \]

Absorption Laws
\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]

Complement Laws
\[ A \cup \overline{A} = U \]
\[ A \cap \overline{A} = \emptyset \]
Set Identities

Proof

Theorem (De Morgan’s Law)

\(\overline{A \cup B} = \overline{A} \cap \overline{B}\)
De Morgan’s Law for Sets.

\[ \forall \text{ sets } A, B \quad \overline{A \cup B} = \overline{A} \cap \overline{B} \]

Given and UI

\[ \{x \mid x \not\in A \cup B\} \]

Def \( \overline{Q} \)

\[ \{x \mid x \not\in \{y \mid y \in A \lor y \in B\}\} \]

Def \( \cup \)

\[ \{x \mid x \in \{y \mid \neg(y \in A \lor y \in B)\}\} \]

Def \( \not\in \)

De Morgan’s

\[ \{x \mid x \not\in A \land x \not\in B\} \]

Simplification/membership

\[ \{x \mid x \not\in A\} \cap \{y \mid y \not\in B\} \]

Def \( \cap \)

\[ \therefore \quad \overline{A} \cap \overline{B} \]

Def \( \overline{Q}, \mathrm{UG} \)
Example

Use the set identities to show that

\[ A \cup (B \cap C) = (\overline{C} \cup \overline{B}) \cap \overline{A} \]
Multi-Union/Intersection

Notation

Definition

The union of $A_1, A_2, ..., A_n$ can be denoted ($\bigcup$)

$$A_1 \cup A_2 \cup ... \cup A_n = \bigcup_{i=1}^{n} A_i$$

The intersection of $A_1, A_2, ..., A_n$ can be denoted ($\bigcap$)

$$A_1 \cap A_2 \cap ... \cap A_n = \bigcap_{i=1}^{n} A_i$$
Multi-Union/Intersection

Example

Let $A_i = \{1, 2, \ldots, i\}$. Give some examples of these sets.
Multi-Union/Intersection

Example

Let $A_i = \{1, 2, \ldots, i\}$.

Give some examples of these sets.

$A_1 = \{1\}$

$A_2 = \{1, 2\}$

$A_5 = \{1, 2, 3, 4, 5\}$

$A_9 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Multi-Union/Intersection

Example

Let $A_i = \{1, 2, \ldots, i\}$.

\[ \bigcup_{i=1}^{\infty} A_i = \]
Multi-Union/Intersection

Example

Let \( A_i = \{1, 2, \ldots, i\} \).

\[
\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+
\]
Multi-Union/Intersection

Example

Let $A_i = \{1, 2, \ldots, i\}$.

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$$

$$\bigcap_{i=1}^{\infty} A_i =$$
Multi-Union/Intersection

Example

Let $A_i = \{1, 2, \ldots, i\}$.

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$$

$$\bigcap_{i=1}^{\infty} A_i = \{1\}$$