Basic Structures

Sequences and Series

February 29, 2016
Outline

1. Homework

2. Sequences
2.3 10 Which of the following functions from \([a, b, c, d]\) to itself are one-to-one. What does 1-to-1 mean?
2.3 10 Which of the following functions from \([a, b, c, d]\) to itself are one-to-one. What does 1-to-1 mean? Each element in the range has a single pre-image in the domain.

- a \(f(a) = b, f(b) = a, f(c) = c, f(d) = d\)
- b \(f(a) = b, f(b) = b, f(c) = d, f(d) = c\)
- c \(f(a) = d, f(b) = b, f(c) = c, f(d) = d\)

PS: Which are onto? What does onto mean?
2.3 10 Which of the following functions from \([a, b, c, d]\) to itself are one-to-one. What does 1-to-1 mean? Each element in the range has a single pre-image in the domain.

a \( f(a) = b, f(b) = a, f(c) = c, f(d) = d \)

b \( f(a) = b, f(b) = b, f(c) = d, f(d) = c \)

c \( f(a) = d, f(b) = b, f(c) = c, f(d) = d \)

PS: Which are onto? What does onto mean? Each element in the codomain is in the range (the codomain is the range).
### Sequence

#### Motivation

A taxi charges $1.00 for the first mile and $0.50 for each additional mile.

So

<table>
<thead>
<tr>
<th>Miles</th>
<th>Cost($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>1.50</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
</tr>
<tr>
<td>4</td>
<td>2.50</td>
</tr>
<tr>
<td></td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$1.00 + 0.50(n - 1)$</td>
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**Sequence**

**Motivation**

**Example**

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Another way to express this:

$C_1 = 1.00$, $C_2 = 1.50$, $C_3 = 2.00$, $C_4 = 2.50$, ...

This is an example of a sequence.
Sequence

Definition

A **sequence** is a function from a subset of the integers (usually \(\{0, 1, 2, \ldots\}\) or the set \(\{1, 2, 3, \ldots\}\)) to a set \(S\).

A sequence, say named \(S\), is denoted by the variable \(S\) or the set of terms, \(\{s_n\}\). The term \(s_n\) is the \(n^{th}\) value in the sequence. Note that \(S\) or \(\{s_n\}\) denotes the entire sequence.

Note in our example: \(C : \mathbb{Z}^+ \rightarrow \mathbb{R}\)  
\[C(1) = C_1 = 1.00, \text{ etc.}\]
Sequence

Definition

Example

Let \( s = 2, 4, 6, \ldots \)
\[ s_1 = 2, s_2 = 4, \ldots, s_n = 2n \]
Sequence

Definition

Example

Let $s = 2, 4, 6, ...$

$s_1 = 2, s_2 = 4, ..., s_n = 2n$

$s$ is an example of an infinite sequence.

Example

$t = a, a, b, a, b$

$t_1 = a, t_2 = a, t_3 = b, t_4 = a, t_5 = b$
**Sequence**

**Definition**

**Example**

Let \( s = 2, 4, 6, \ldots \)
\( s_1 = 2, s_2 = 4, \ldots, s_n = 2n \)

\( s \) is an example of an infinite sequence.

**Example**

\( t = a, a, b, a, b \)
\( t_1 = a, t_2 = a, t_3 = b, t_4 = a, t_5 = b \)

\( t \) is an example of a finite sequence.
Sequence
Geometric Progression

Definition
A geometric progression is a sequence of the form

\[ \{ g_n \}_{n=0}^\infty = a, ar, ar^2, ..., ar^n, ... \]

where \( a \) is the initial term and \( r \) is the common ratio, and \( a \) and \( r \) are real numbers and \( g_n = ar^n \).

Example
\[ \{ b_n \}_{n=0}^\infty \quad b_n = (-1)^n \]
Sequences

Sequence

Geometric Progression

Definition

A geometric progression is a sequence of the form

$$\{g_n\}_{n=0}^{\infty} = a, ar, ar^2, \ldots, ar^n, \ldots$$

where $a$ is the initial term and $r$ is the common ratio, and $a$ and $r$ are real numbers and $g_n = ar^n$.

Example

$$\{b_n\}_{n=0}^{\infty} \quad b_n = (-1)^n$$

$b_0 = 1, b_1 = -1, b_2 = 1, \ldots$
**Definition**

A geometric progression is a sequence of the form

\[ \{g_n\}_{n=0}^{\infty} = a, ar, ar^2, \ldots, ar^n, \ldots \]

where \(a\) is the initial term and \(r\) is the common ratio, and \(a\) and \(r\) are real numbers and \(g_n = ar^n\).

**Example**

\[ \{b_n\}_{n=0}^{\infty} \quad b_n = (-1)^n \]

\(b_0 = 1, b_1 = -1, b_2 = 1, \ldots\)

Is a geometric progression. Initial term = ?, common ratio = ?
Sequence
Geometric Sequence Practice

Example

\[
\{c_n\}_{n=0}^{\infty} \quad c_n = 2 \cdot 5^n \\
c_0 = 2, \quad c_1 = 10, \quad c_2 = 50, \quad c_3 = 250, \quad c_4 = 1250, \ldots
\]

Initial term? Ratio?
**Sequence**

**Geometric Sequence Practice**

**Example**

\[
\{c_n\}_{n=0}^\infty = 2 \cdot 5^n \\
\]

\[
c_0 = 2, \ c_1 = 10, \ c_2 = 50, \ c_3 = 250, \ c_4 = 1250, \ldots \\
\]

Initial term? Ratio?

**Example**

\[
\{d_n\}_{n=0}^\infty = 6 \cdot (1/3)^n \\
\]

\[
d_0 = 6, \ d_1 = 2, \ d_2 = \frac{2}{3}, \ d_3 = \frac{2}{9}, \ d_4 = \frac{2}{27}, \ldots \\
\]

Initial term? Ratio?
Sequence
Arithmetic Sequence

Definition
An arithmetic progression is a sequence of the form

\[ \{s_n\}_{n=0}^\infty = a, a+d, a+2d, a+3d, ..., a+nd, ... \]

where \( a \) is the initial term and \( d \) is the common difference, and \( a \) and \( d \)
are real numbers and \( s_n = a + nd \).
Sequences

Sequence

Arithmetic Sequence

Example

\( \{ s_n \}_{n=0}^{\infty} = -1 + 4n \)

\( s_0 = -1, s_1 = 3, s_2 = 7, s_3 = 11, \ldots \)

Is an arithmetic progression with ???
Sequence

Arithmetic Sequence

Example

\[ \{s_n\}_{n=0}^{\infty} = -1 + 4n \]
\[ s_0 = -1, s_1 = 3, s_2 = 7, s_3 = 11, \ldots \]
Is an arithmetic progression with ???

Example

\[ \{t_n\}_{n=0}^{\infty} = 7 - 3n \]
\[ t_0 = 7, t_1 = 4, t_2 = 1, t_3 = -2, \ldots \] Is an arithmetic progression with ???
Example

Find a formula for the following sequence:

\[ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \]
Sequence

Practice

Example
Find a formula for the following sequence:

$$1, 1/2, 1/4, 1/8, 1/16, ...$$

Example
Find a formula for the following sequence:

$$1, 3, 5, 7, 9, ...$$
Sequence

Practice

Example
Find a formula for the following sequence:

1, 1/2, 1/4, 1/8, 1/16, ...

Example
Find a formula for the following sequence:

1, 3, 5, 7, 9, ...

Example
Find a formula for the following sequence:

1, −1, 1, −1, ...
Sequence

Practice

Example

Find a formula for the following sequence:

5, 11, 17, 23, 29, 35, 41, ...
Sequence

Practice

Example
Find a formula for the following sequence:

\[ 5, 11, 17, 23, 29, 35, 41, \ldots \]

Example
Find a formula for the following sequence:

\[ 1, 7, 25, 79, 241, 727, 2185, 6559, \ldots \]
Example

Find a formula for the following sequence:

5, 11, 17, 23, 29, 35, 41, ...

Example

Find a formula for the following sequence:

1, 7, 25, 79, 241, 727, 2185, 6559, ...

Example

Find a formula for the following sequence:

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \]
**Sequence**

**Harmonic Sequence**

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**Example**

Consider the sequence \( \{ a_n \}_{n=1}^{\infty} = \frac{1}{n} \).

\[ a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \ldots \]

This is known as a **harmonic sequence**.
Definition

A series or summation is the sum of the elements of a sequence.

\[ \sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n \]

The variable \( i \) is called the index of summation with inclusive lower limit \( m \) and upper limit \( n \).
Example

Express the sum of the first 100 terms of the sequence \( \{a_n\} \) where \( a_n = 1/n \) for \( n = 1, 2, 3, \ldots \).
Example

Express the sum of the first 100 terms of the sequence \( \{a_n\} \) where 
\[ a_n = \frac{1}{n} \quad \text{for} \quad n = 1, 2, 3, \ldots. \]

Example

What is the value of 
\[ \sum_{j=1}^{5} j^2. \]
Example

Suppose we have the sum $\sum_{i=1}^{5} i^2$. Change the index of summation to be 0 to 4 instead of 1 to 5.
Series

Geometric Series Closed Form

Theorem

If $a$ and $r$ are real numbers and $r \neq 0$, then

$$\sum_{i=0}^{n} ar^i = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$
## Series

### Useful Closed Forms

#### Example

<table>
<thead>
<tr>
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</tr>
<tr>
<td>$\sum_{i=1}^{n} i$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
<tr>
<td>$\sum_{i=1}^{n} i^2$</td>
<td>$\frac{2n(n+1)(2n+1)}{6}$</td>
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**Series**

**Useful Closed Forms**

**Example**

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<td>$\sum_{i=1}^{n} i^2$</td>
<td>$\frac{n(n+1)(2n+1)}{6}$</td>
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**Example**

Find

$$\sum_{k=50}^{100} k^2$$