Basic Structures
Cardinality

March 14, 2016
Outline

1. Cardinality
Definition
Sets $A$ and $B$ have the same cardinality if and only if there is a one-to-one correspondence from $A$ to $B$. Written $|A| = |B|$. If there is a one-to-one function from $A$ to $B$, $|A| \leq |B|$; when $|A| \leq |B|$ and $A$ and $B$ have different cardinality, $|A| < |B|$.
Cardinality of a Set

Example (Finite Sets)

\[ \{ \text{A, E, I, O, U} \} \]

\[ \{ 0, 1, 2, 3, 4 \} \]
Cardinality of a Set

Example (Finite Sets)

\[
\{ \text{A, E, I, O, U} \} \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
\{ 0, 1, 2, 3, 4 \}
\]
Cardinality of a Set

Example (Infinite Sets)

\[
\begin{align*}
\{ & 1 & 2 & 3 & 4 & 5 & \ldots \} & \mathbb{Z}^+ \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
\{ & 1 & 3 & 5 & 7 & 9 & \ldots \} & \text{odd}(\mathbb{Z}^+) 
\end{align*}
\]
Countable

Definition
A set that is either finite or has the same cardinality as \( \mathbb{Z}^+ \) is said to be countable. A set that is not countable is said to be uncountable.

An infinite countable set, \( S \), has cardinality of \( \aleph_0 \) (Hebrew letter aleph). \( |S| = \aleph_0 \) read “\( S \) has cardinality aleph naught (or aleph null).”
Hilbert’s Grand Hotel

Consider a Grand Hotel with $\aleph_0$ rooms. (What is another way to say this?)
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Hilbert’s Grand Hotel

Consider a Grand Hotel with $\aleph_0$ rooms. (What is another way to say this?) The rooms in the Grand Hotel can be numbered 1, 2, 3, ... (Why?) Assume all rooms are occupied and a weary traveler arrives. Hilbert claims the Grand Hotel always has room for one more. How?
Countably Infinite Sets

Example
Show that $\mathbb{Z}$ is countably infinite.
Countably Infinite Sets

Example
Show that $\mathbb{Z}$ is countably infinite.

Example
Show that $\mathbb{Q}^+$ is countably infinite.
An Uncountable Infinity

Theorem

Show that $\mathbb{R}$ is uncountable.

Example

How could you approach this problem?
An Uncountable Infinity

Theorem

*Show that \( \mathbb{R} \) is uncountable.*

Example

How could you approach this problem?
How would you approach a problem to show something is irrational?
Results

Theorem

*If* $A$ and $B$ *are countable sets, then* $A \cup B$ *is also countable.*
**Results**

**Theorem**

*If $A$ and $B$ are countable sets, then $A \cup B$ is also countable.*

**Proof.**

Given $A$ and $B$, countable sets. WLOG, $A$ and $B$ are disjoint. Also WLOG, if exactly one set is finite, set $B$ is finite. 

Cases:

i. $A$ and $B$ finite: $A \cup B$ is finite (thus countable).

ii. Only $A$ infinite: List $B$ followed by $A$ and you can match every entry in $A \cup B$ with a positive integer.

iii. Both infinite: Alternate elements from $A$ and $B$. This sequence can be paired with $\mathbb{Z}^+$ and is countable.

By showing $A \cup B$ is countable in every possible case, proves that if $A$ and $B$ are countable, $A \cup B$ is countable.
Schröder-Bernstein Theorem

Theorem

If $A$ and $B$ are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

If there is a one-to-one function $f : A \to B$ and a one-to-one function $g : B \to A$, then there is a one-to-one correspondence between $A$ and $B$. 
Computable

Definition
A function is *computable* if there is a computer program in some programming language that finds the values of the function. If a function is not computable, it is *uncomputable*. 