Induction and Recursion

Strong Induction

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Prove that 6 divides $n^3 - n$, $n \in \mathbb{Z}$. 

5.1 34
2.6 27 Let \( A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) \( B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \)

Find

a. \( A \lor B \)

b. \( A \land B \)

c. \( A \oplus B \)

3.1 10 Devise an algorithm to compute \( x^n \) where \( x \) is a real number and \( n \) is an integer. (Hint: First give a procedure for computing \( x^n \) when \( n \) is nonnegative by successive multiplication by \( x \), starting with 1. Then extend this procedure, and use the fact that \( x^{-n} = \frac{1}{x^n} \) to compute \( x^n \) when \( n \) is negative.)
Strong Induction

Motivation

Example

Prove that any postage of 4¢ or more can be made using only 2¢ and 5¢ stamps.
Strong Induction

Motivation

Example

Prove that any postage of 4¢ or more can be made using only 2¢ and 5¢ stamps.

Proof.

Basis step: $n = 4¢$: use 2 2¢ stamps.

Inductive step: Assume truth of statement for $n \geq 4$: $n¢$ postage can be made using only a mix of 2¢ and 5¢ stamps.

Show that this implies we can make postage for $n + 1¢$. 

\[ \square \]
Strong Induction

Principle of Strong Induction

Definition

To prove $P(n)$ true for all positive integers $n$:

Basis step  Show that $P(1)$ is true.

Inductive step  Show that $P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(n) \rightarrow P(n + 1)$ is true for all positive integers $n$. 
Strong Induction

Principle of Strong Induction

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State regular induction in the same form?
Strong Induction

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Strong Induction

Example: Induction

Example
Show that postage of 12 cents or more can be achieved using only 4-cent and 5-cent stamps.

Using induction.
Basis step: 12 cents: use 3 4-cent stamps;
13 cents: use 2 4-cent stamps and 1 5-cent stamp;
Induction step: Assume you can do it for \( n \) cents. Show that implies it can be done for \( n + 1 \) cents.
If the solution for \( n \) cents includes a 4-cent stamp, remove 1 4-cent stamp, replacing it with a 5-cent stamp (total is \( n + 1 \)).
Otherwise: \( n = 5k, \ k \geq 3 \). Remove 3 5-cent stamps and add back 4 4-cent stamps. Sum is reduced by 15 and increased by 16 cents yielding \( n + 1 \) cents.
Strong Induction

Example: Strong Induction

Example

Show that postage of 12 cents or more can be achieved using only 4-cent and 5-cent stamps.

Using \textit{strong induction}.

Basis step: 12 cents: use 3 4-cent stamps;
13 cents: use 2 4-cent stamps and 1 5-cent stamp;
14 cents: use 1 4-cent stamp and 2 5-cent stamps
15 cents: use 0 4-cent stamps and 3 5-cent stamps.

Induction step: Show $P(j)$,
$12 \leq j \leq n \land n \geq 15 \rightarrow P(n + 1)$
$n \geq 15 \rightarrow P(n - 3)$

Use solution for $n - 3$ and add 1 4-cent stamp to yield $n + 1$ cents in postage.
Strong Induction

Recursion and Strong Induction

Strong induction is handy when a sequence is defined recursively.

Example

Let $c_1 = 0$ and $c_n = c_{\lfloor n/2 \rfloor} + n$. Show that $c_n < 4n$ for all $n > 1$. 
Example

Consider the product: $a_1a_2 \ldots a_n$. Prove that if we insert parentheses in any manner whatsoever to multiply the numbers $a_1, a_2, \ldots a_n$, we perform $n - 1$ multiplications.
Well-Ordering

Definition

Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.
Example

Use the well-ordering property to prove the division algorithm. Recall the division algorithm: If $n$ is an integer and $d$ is a positive integer, then there are unique integers $q$ and $r$ such that

$$n = dq + r, \quad 0 \leq r < d$$