1. Suppose \( a \) is some value, \( a > 0 \), and that \( x \) is an approximation to \( a \), with \( x > 0 \). Assume that the absolute error of \( x \) as an approximation to \( a \) is less than \( \varepsilon \) for some \( \varepsilon > 0 \). That is, \( |x - a| < \varepsilon \). For \( \varepsilon \) sufficiently small, we can further assume that \( \varepsilon < a \). Under these assumptions, prove that \( \varepsilon a(a - \varepsilon) \) is an upper bound on the absolute error of \( 1/x \) as an approximation to \( 1/a \). (Hint: first show that \( 0 < a - \varepsilon < x \).)

2. Suppose \( x \) is an approximation to 100 and that the absolute error of this approximation is less than \( 1/10 \). Use the previous problem to give an upper bound on the absolute error of \( 1/x \) as an approximation to \( 1/100 \).

3. Repeat problem 1, except assume that the relative error of \( x \) as an approximation to \( a \) is less than \( \varepsilon \) and that \( \varepsilon < 1 \). Prove that \( \frac{\varepsilon}{1 - \varepsilon} \) is an upper bound on the relative error of \( 1/x \) as an approximation to \( 1/a \). (Hint: first show that \( 0 < a(1 - \varepsilon) < x \).)

4. Suppose \( a \) is some value, \( a \neq 0 \), and that \( x \) is an approximation to \( a \). Assume that the relative error of \( x \) as an approximation to \( a \) is less than \( \varepsilon \) for some \( \varepsilon > 0 \). For any nonzero value \( k \), prove that the relative error of \( kx \) as an approximation to \( ka \) is less than \( \varepsilon \).

5. Is the previous statement true if “relative error” is replaced by “absolute error”? If so, prove it. If not, give a counterexample.

6. Suppose we have some method to compute approximations to some unknown value \( a \), where \( a > 0 \), and that \( x \) is such an approximation, with \( x > 0 \). Assume that we have an upper bound \( \varepsilon \) on the absolute error of this approximation for some \( \varepsilon > 0 \). Getting an upper bound on the relative error \( \frac{|x - a|}{a} \) requires knowing the value of \( a \), which we obviously don’t know. Clearly \( \frac{|x - a|}{a} \) is bounded by \( \frac{\varepsilon}{a} \). However, if \( x \) is an approximation to \( a \), then \( \frac{\varepsilon}{x} \) should be an approximate upper bound. Indeed, show that if \( \varepsilon \) is sufficiently small such that \( \varepsilon < \frac{|x|}{2} \), then the relative error \( \frac{|x - a|}{a} \) is bounded by \( \frac{2\varepsilon}{x} \).

7. Use the bisection method to find a root of \( f(x) = x^5 + x + 1 \). First, plot this function on the interval \([-2, 2]\) to convince yourself that this function has a root near \(-1\). Choose appropriate endpoints of an interval that traps the root and such that the function values have opposite signs at these endpoints. Run the Bisection method (Bisection.m in the class Code directory) on this function to determine the value of the root with error not to exceed \( 0.5 \times 10^{-6} \). Report on the endpoints of the interval you chose, the value of the root, and the number of iterations Bisection took to evaluate this root.

8. Find the root of the function \( f(x) \) in the previous problem using Newton’s method (Newton.m in the class Code directory), using the same error bound of \( 0.5 \times 10^{-6} \). Report on the initial value you chose to start the iterations, the value of the root, and the number of iterations.